A knot theory of physics, spacetime in co-dimension 2

C. Ellgen*

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Abstract

Spacetime is assumed to be a 4-dimensional manifold embedded in a 6-dimensional Minkowski space. The spacetime manifold can be knotted and those knots correspond to particles. Distortion of the spacetime manifold around knots generates fields. Gravity, electrodynamics, and quantum properties result from the assumptions.

*Electronic address: cellgen@gmail.com; www.knotphysics.net

I. INTRODUCTION

A. Purpose

The purpose of this paper is to describe physics in terms of a single mathematical object, the spacetime manifold. The results of quantum field theory have already provided outstanding mathematical accuracy in predicting physical phenomena. This paper does not attempt to argue with the technique or formulation of quantum field theory. The purpose instead is to answer questions such as "what is a field?", "what are mass and charge?", and even "what is a particle?". With these answers we will derive some principal results of physics.

B. Motivation

What line of reasoning leads to this theory? General relativity presumes that gravity is curvature of the spacetime manifold. Because the gravitational field is generated by mass, the next step is to describe mass in a way that leads to spacetime curvature. Next one would describe particles in a way that gives them mass. Given that description of particles, do any other fields arise without further modifications to the theory? Finally, how do the results of quantum mechanics follow?

The first step is to describe mass. If mass generates spacetime curvature then perhaps mass *is* spacetime curvature. The equations of general relativity dictate that spacetime curvature propagates dispersively in space, in contrast to the nature of particles. To prevent that dispersion, the spacetime curvature of a particle could be restricted by topological constraints. Knots suit this constraint. Knots can only exist on an n-dimensional manifold embedded in an n+2-dimensional space. If the space is n+3-dimensional or higher, the knots spontaneously untie. If the space is n+1-dimensional or lower, the manifold cannot form a knot. Therefore a 4-dimensional spacetime manifold with knots is necessarily embedded in a 6-dimensional space. The rest of the paper will develop the notion of particles as knots, showing how that assumption gives rise to the known particles, fields, quantum mechanical results, and other aspects of physics.

C. Particles are knots in spacetime

The spacetime manifold is a 4-dimensional manifold in a Minkowski 6-space and we require that the spacetime manifold cannot self-intersect. These two conditions allow the manifold to be knotted. The most natural method of producing knots in the manifold is cobordism. However, if the manifold is everywhere Lorentzian then any cobordism is also a diffeomorphism, which implies that no topology change is possible. While we have assumed that the 6-space is Minkowski, the manifold is embedded in the 6-space and its metric is inherited, allowing for degeneracy. This distinction is the way that topology change occurs on M. We address this is greater detail in "The particles and topology of knot physics"[1].

II. THE SPACETIME MANIFOLD

A. Transformations and metrics of the space

The spacetime manifold, M, is embedded in a 6-dimensional Minkowski space, Ω. The metric $\eta^{\mu\nu}$ on Ω is $diag(1,-1,-1,-1,-1,-1)$. The Lorentz transformations of Ω are $O(1,5)$. The space, Ω, can be transformed by any Lorentz transformation and the manifold M will be transformed in the natural way. If a point p of M is at coordinates x^μ in Ω then a Lorentz transformation Λ of Ω acts on p and x^μ in the same way, $\Lambda p = \Lambda x$. Any equation of physics must be invariant under these transformations.

The metric on Ω is $\eta^{\mu\nu}$ but we also need a metric on our manifold. For that purpose we choose $\bar{\eta}^{\mu\nu}$, which is the restriction of $\eta^{\mu\nu}$ to the manifold M. The product $\bar{\eta}^{\mu\nu} v_\mu w_\nu$ is the value of $\eta^{\mu\nu}$ on the projections of the vectors v_μ and w_ν onto M.

We retain the coordinate notation with x^μ and x_μ contravariant and covariant tensors with six coordinates. Tensorial transformations and contractions of any number of contravariant and covariant indices are the same with the exception of an additional two spacelike components as determined by $\eta^{\mu\nu}$.

For convenience we choose our coordinates so that flat spacetime is parallel to the x_0, x_1, x_2, x_3 axes.

B. Differentiation

The derivatives of tensors on M need to produce the appropriate resulting tensor. For a scalar field f on M we define $\partial^\mu f$ such that the derivative in the i direction is equal to the component of the gradient of f in the i direction. The derivatives are taken with respect to the metric $\eta^{\mu\nu}$. The vector field $\partial^\mu f$ is the gradient of f on M. Derivatives ∂^μ and ∂_μ of scalar fields on M will always be vectors (and covectors) in the tangent space of the manifold. The derivative ∂^μ will sometimes be denoted by a comma: $\partial^\mu f = f^{,\mu}$. Derivatives of higher rank tensors are done by differentiating each component as if it were a scalar field. These derivatives are taken with respect to distances in Minkowski space, which is flat. Therefore we use ordinary partial derivatives. Because the derivative ∂^μ does not require any parallel translation of vector fields, the derivatives commute.

Does general covariance apply to knot physics? In knot physics we assign coordinates by their position in the Minkowski 6-space. Alternatively, we could assign coordinates on the manifold using a mapping from $\mathbf{R}^4$ to the manifold. Distances on the manifold would be described by the appropriate metric. Then the only information available is the geodesic distance between points on the manifold. We find that general covariance does apply for our description of gravity, electromagnetism, and a variety of other applications. However, particle knots require that we know how the manifold is constrained by non-self-intersection, which requires knowledge of the geometry of the manifold in the 6-space.

C. Potential

We define a potential function A^ν on M. *The potential function in spacetime with no electroweak field is* $A^\nu = x^\nu$. (The potential is equal to the coordinate.) This is an important difference from the usual assumption that the potential is zero if there is no field. Given that $A^\nu = x^\nu$ if there is no field, we assume that A^ν converges to x^ν as we go to infinite distance.

As part of the definition of A^ν we require

$$\partial_\nu A^\nu = -2. \tag{1}$$

D. The field equation

Define the tensor $F^{\mu\nu}$

$$F^{\mu\nu} = A^{\nu,\mu} - A^{\mu,\nu} \tag{2}$$

similarly to Maxwell's electromagnetism tensor. We choose a Lagrangian density

$$L = \sigma + F^{\mu\nu} F_{\mu\nu} \tag{3}$$

for some constant $\sigma < 0$. Then the action is

$$S[M] = \int_M \sigma + F^{\mu\nu} F_{\mu\nu} dM \tag{4}$$

Here dM is the natural measure on the manifold, the 4-volume in the Minkowski space.

Later we will modify our assumptions in order to derive quantum properties. However, for the moment we are done with our assumptions.

If the manifold is flat then we can find the field equation in the way familiar from electromagnetism:

$$\partial^\mu \frac{\partial L}{\partial A^\nu{}_{,\mu}} - \frac{\partial L}{\partial A^\nu} = 0, \tag{5}$$

which implies

$$\partial_\mu F^{\mu\nu} = 0 \tag{6}$$

If the manifold is not flat but $A^\nu = x^\nu$ then the action is $S[M] = \int_M \sigma dM$, which implies that the Lagrangian minimizes the volume of the manifold. In section VII we will show that this action implies the action of general relativity.

E. Energy and momentum

We want to derive the energy-momentum tensor $T^{\mu\nu}$ at a point p, assuming that the tangent space at p is parallel to the span of x_0, x_1, x_2, x_3. The tensor $T^{\mu\nu}$ depends only on first derivatives of the A^ν potential. Therefore, if we can determine $T^{\mu\nu}$ when M lies in a four-dimensional linear subspace at a point p, then we have derived $T^{\mu\nu}$ for any geometry that has the same tangent space at p. Likewise, if we use only tensors in the definition of

$T^{\mu\nu}$ then the Lorentz invariance of tensors guarantees that the definition of $T^{\mu\nu}$ holds for any geometry and any orientation of the tangent space at the point of evaluation.

To get a tensor $T^{\mu\nu}$ that satisfies $\partial_\mu T^{\mu\nu} = 0$,

$$T^{\mu\nu} = -A^{\alpha,\mu}\frac{\partial L}{\partial A^{\alpha}_{,\nu}} + \bar{\eta}^{\mu\nu} L \tag{7}$$

therefore

$$T^{\mu\nu} = -A^{\alpha,\mu}\frac{\partial(\sigma + F^{\mu\nu}F_{\mu\nu})}{\partial A^{\alpha}_{,\nu}} + \bar{\eta}^{\mu\nu}(\sigma + F^{\mu\nu}F_{\mu\nu}) \tag{8}$$

$$T^{\mu\nu} = 4A^{\alpha,\mu}F^{\nu}_{\ \alpha} + \bar{\eta}^{\mu\nu}(\sigma + F^{\mu\nu}F_{\mu\nu}) \tag{9}$$

which satisfies $\partial_\mu T^{\mu\nu} = 0$, but $T^{\mu\nu}$ is not symmetric. To symmetrize we add a term $-4A^{\mu,\alpha}F^{\nu}_{\ \alpha}$ to get

$$T^{\mu\nu} = -4F^{\alpha\mu}F^{\nu}_{\ \alpha} + \bar{\eta}^{\mu\nu}(\sigma + F^{\mu\nu}F_{\mu\nu}) \tag{10}$$

which is symmetric and satisfies $\partial_\mu T^{\mu\nu} = 0$. All of this is familiar from the development of electromagnetism. The only change is the addition of the σ scalar, which propagates through the equations.

Finally, we need to integrate $T^{\mu\nu}$ to get the energy and momentum. Integrating over M requires a measure that accounts for the possibility that the manifold is moving. We use a unit of volume dM on M. If the surface is at rest (or, equivalently, is moving parallel to itself) then the integrating volume is $dM = dtdV$. However, if the manifold is in motion perpendicular to itself with velocity β then the volume changes according to the metric and becomes $dM = (\sqrt{1-\beta^2})dtdV$. Using dM as a unit of volume of M and incorporating $\sqrt{1-\beta^2}$ into the measure gives

$$\int T^{\mu\nu}dM = \int [-4F^{\alpha\mu}F^{\nu}_{\ \alpha} + \bar{\eta}^{\mu\nu}(\sigma + F^{\mu\nu}F_{\mu\nu})](\sqrt{1-\beta^2})dtdV. \tag{11}$$

To find the energy, for example, we can compare integrals over the whole manifold

$$\int Edt = \int T^{00}dM = \int T^{00}(\sqrt{1-\beta^2})dtdV \tag{12}$$

Thus, on a small region dV, the energy is $E = T^{00}(\sqrt{1-\beta^2})dV$.

F. Rest mass

To find the energy of a particle we will begin by assuming that the particle has no fields, i.e. $A^\nu = x^\nu$; the potential is equal to the position in six-space. Then $T^{\mu\nu}$ at any point on the particle is the Lorentz transformation of $T^{\mu\nu}$ for flat space. In flat space with no fields $T^{\mu\nu} = \sigma\bar{\eta}^{\mu\nu} = \sigma \times diag(1,-1,-1,-1,0,0)$. If the manifold has no fields but is in motion with velocity $\vec{\beta} = (1,0,0,0,-\beta,0)$ then

$$T^{\mu\nu}dM = \sigma\sqrt{1-\beta^2}\bar{\eta}^{\mu\nu}dtdV = \sigma(1/\gamma)\bar{\eta}^{\mu\nu}dtdV \tag{13}$$

$$T^{\mu\nu}dM = \sigma/\gamma \begin{bmatrix} \gamma^2 & 0 & 0 & 0 & \beta\gamma^2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta\gamma^2 & 0 & 0 & 0 & \beta^2\gamma^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} dtdV. \tag{14}$$

For a volume V of spacetime without fields, the energy is proportional to γV and the momentum is proportional to $\beta\gamma V$ in the direction of motion. Then the volume V has the characteristics of rest mass. However, this is not the entirety of a particle's rest mass. Fields and rotational momenta also contribute to the masses of particles.

III. THE ELECTROMAGNETIC FIELD

A. Charge

We would like to reproduce electromagnetism. Spacetime has no electromagnetic fields if $A^\nu = x^\nu$. Therefore we need a spacetime topology that requires $A^\nu \neq x^\nu$. We discuss this topology in [1]. For now we assume that a charged particle is a knot that constrains the potential. If the particle is at rest, then we assume that the knot produces a constant divergence of the A^0 gradient. If the charge is in motion then Lorentz invariance implies that other components of the A^ν field are affected.

B. Field generation

Flat spacetime satisfies $0 = \partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$. Using $\partial_\nu A^\nu = -2$, this implies $\partial_\mu \partial^\mu A^\nu = 0$. A charged particle is a knot with a topological condition on the A^ν field. The field condition is a singularity of $\partial^\mu \partial_\mu A^\nu$. We can approximate the divergence of the A^0 field with the Dirac delta. For the case of a knot at rest at the origin,

$$\partial^\mu \partial_\mu A^0 = \delta(\vec{x}) \tag{15}$$

(Here δ is the Dirac delta function and $\vec{x}$ refers only to the spacelike coordinates.) Because A^ν transforms tensorially, we need a quantity on the right side of the equation that likewise transforms tensorially. We therefore invent a quantity j^ν which takes the value $j_{rest}(\vec{x}) = (\delta(\vec{x}), 0, 0, 0, 0, 0)$ when the charge is at rest. Clearly there is a unique tensor j^ν taking the velocity vector of the particle at rest to the velocity vector of the particle in motion. Then j^ν satisfies

$$\partial^\mu \partial_\mu A^\nu = j^\nu. \tag{16}$$

We use j^ν as a method of counting singularities so that one need not bother with the geometry of the knot.

To reproduce Maxwell's equations, we assume M is flat, in the span of x^0, x^1, x^2, x^3, and only the potentials A^0, A^1, A^2, A^3 are nonzero. Then we reproduce the potential of Maxwell's equations by taking $A^\nu - x^\nu$. We have $\partial_\nu (A^\nu - x^\nu) = -2 - (-2) = 0$ and this is the Lorentz gauge condition for Maxwell's equation. Likewise, Maxwell's equation is

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 4\pi j^\nu \tag{17}$$

For flat space this is the same as our equation $\partial_\mu F^{\mu\nu} = j^\nu$, assuming we absorb the 4π into our definition of j^ν.

IV. SPIN

Spin 1/2 particles have their own quantum statistics. We assume that the topology of a spin 1/2 particle has a non-orientable spacelike slice. A spacelike 3-dimensional submanifold D passing through a spin 1/2 particle reverses the orientation of the coordinate frame tangent to D. All such orientation changes are equivalent to reflections; some vector in the coordinate frame maps to its own negative.

Because the particle reverses the orientation of the coordinate frame, there is no 4-dimensional basis that is consistent with the geometry. We fix this using a double-cover. We change from vectors in $SO(3,1)$ to spinors in $Spin(3,1)$. To change basis we can use the Dirac matrices γ^μ, as in the relativistic quantum theory of fermions.

If the coordinate frame of the three spacelike vectors changes orientation when it is translated through the knot then we say it has spin $n/2$ for n an odd integer. The topology and spin angular momentum of fermions will be discussed in [1]

V. THE ELECTROWEAK FORCE

In electromagnetism we described the A^ν field if the manifold is flat. How can we describe the A^ν field on M, if we do not assume that M is flat?

If M is flat then there are Lorentzian velocity transformations such that the manifold is everywhere parallel to the velocity. These velocities are in the tangent space of the manifold, which for convenience we assume is in the span of x^0, x^1, x^2, x^3. Using these transformations on a flat manifold, the A^ν field transforms as if the field were massless. If M is not flat then the manifold may not be parallel to the velocity used in the Lorentzian transformation. From section II F we see that this would cause the field energy to transform as if it has rest mass.

If M is not flat then the tangent space is not constant. Finding derivatives of A^ν requires taking derivatives in the tangent space to M. The derivatives $\partial^\mu A^\nu$ determine the change in the A^ν potential along six directions. However, the A^ν potential is only defined on a 4-dimensional manifold. To describe $F^{\mu\nu}$ on arbitrary geometry we will use $SO(6)$ and then use a gauge group to disregard the irrelevant directions.

The tensor $F^{\mu\nu}$ is anti-symmetric and therefore the matrix exponential of $F^{\mu\nu}$ (using ordinary matrix multiplication) is a rotation matrix, $\exp(F^{\mu\nu}) = \Phi^{\mu\nu} \in SO(6)$. For small rotations there is the approximation $\Phi^{\mu\nu} = I^{\mu\nu} + F^{\mu\nu}$ with $I^{\mu\nu}$ the 6×6 identity matrix. However, $\Phi^{\mu\nu} \in SO(6)$ gives us more information than we need at any given point. In electromagnetism M is flat and spanned by x_0, x_1, x_2, x_3 so there is a subgroup $SO(2) \subset SO(6)$ that acts on the vectors in the x_4 and x_5 directions and has no effect on the $\partial^\mu A^\nu$

derivatives. In general, any particular direction of the manifold might be excluded from the tangent space except for the x_0 direction; the manifold always moves in the time direction.

If the manifold is not flat then we need a description of A^ν that extends the electromagnetism description. The new field description has a component that ignores x_4 and x_5 and has gauge group $SO(2)$. There is another component that ignores x_1, x_2, x_3, which would give a gauge group $SO(3)$. However, we saw in section IV that $SO(3)$ does not describe the geometry for spin 1/2 particles because they are non-orientable. To describe the non-orientable geometry we use $SU(2)$, the double cover of $SO(3)$. This gives a combined gauge group of $SU(2) \times SO(2)$.

We can compare to the electroweak theory. Far from particles the manifold is flat, the A^ν field is massless, and it has gauge group $SO(2) \cong U(1)$, which matches electromagnetism in the electroweak theory. Close to particles the manifold is not flat, the A^ν field can have mass, and it has gauge group $SU(2) \times SO(2) \cong SU(2) \times U(1)$, which matches the electroweak unification.[2–4]

The particle decays associated with the weak force will be discussed in [1].

VI. THE STRONG FORCE

The strong force is a consequence of the non-self-intersection constraint of the manifold, specifically quarks. In [1] we describe the topology of quarks. For the moment we only assume that quarks are the components of a knot and that they cannot be separated from each other. The constraint holding them together is topological: the manifold cannot self-intersect. The relative location of the quarks is irrelevant, the important thing is that they are close enough to remain linked.

A. A gauge group for quarks

The location of the quarks is irrelevant as long as they remain linked. To account for the arbitrary location we incorporate a gauge group. For three quarks we have three locations that are vectors in the 5-space. Label those vectors q_j^n where n identifies which quark and j is the index of the 5-vector. For these three vectors we add a non-physical sixth coordinate q_6^n that *does not* correspond to a spacetime coordinate.

It remains to choose the location of the origin of coordinates for the q_j^n vectors and the q_6^n coordinates for each quark. We choose the origin and the q_6^n coordinates so that the q_j^n are all unit vectors and $\sum_n q_j^n = 0$ for all j. If O_j is the origin then $O_j = (\sum_n q_j^n)/3$. We then subtract O_j from each q_j^n. Then each q_6^n is determined by the unit vector constraint. The quarks are constrained to be close to each other. If the quarks are distant then no choice of origin will allow the q_j^n to all be unit vectors. However, the quarks can be arbitrarily close by making the non-physical q_6^n close to one. Five coordinates of the origin and three q_6^n coordinates are 8 variables and there are 9 constraints, therefore we are overconstrained. If we allow one of the q_5^n to also be variable then there is a solution if the quarks are close enough.

As previously mentioned, we can place the quarks anywhere close to each other. Because the placement is arbitrary, we want a gauge group that can map between any two arrangements of the quarks. We represent the quark vectors as complex 3-vectors $q_j^n = (q_1^n + iq_2^n, q_3^n + iq_4^n, q_5^n + iq_6^n)$. We use a matrix $U \in SU(3)$ to map to new quark vectors $q_{new}^n = Uq_{old}^n$. This will preserve the condition of unit magnitude and summing to zero. However, the allowable q_j^n vectors have 18 variables and 9 constraints for 9 degrees of freedom, as opposed to the 8 dimensional group $SU(3)$. Lower dimension implies the elements of $SU(3)$ map to a subset of the possible vectors. Although $SU(3)$ works as a gauge group, it is a subgroup of the gauge group that makes all q_j^n possibilities equivalent.

If we call q_j^n the color charge and use $SU(3)$ as the gauge group then we have a geometric model that matches quantum chromodynamics.

B. Quark properties

Attempting to remove a quark from a particle is equivalent to stretching the knot. The quark cannot be removed and the energy required to stretch the knot is a divergent function of the distance to which it is stretched. This is quark confinement [5].

As the knot shrinks, the Lagrangian dictates the force with which the knot contracts. However, below a certain threshold quantum phenomena become the dominant factor in determining the force of contraction. We see in section VIII that quantum phenomena result from relaxing the condition of the Lagrangian and that relaxation implies the force of contraction goes to zero as the radius becomes small. This is quark asymptotic freedom [6, 7].

VII. GRAVITY

Our assumptions generate geometric consequences whose effect is gravitation.

A. Einstein's field equation

Using the Ricci tensor $R^{\mu\nu}$, Einstein's field equation is the following:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi T^{\mu\nu}. \tag{18}$$

However, this does not match our definitions. The first problem is that the energy tensor $T^{\mu\nu}$ in Einstein's field equation is the energy-momentum tensor of the gravitating source but it does not include the energy-momentum of the gravitational field or flat space. The energy-momentum tensor that we have been using includes the gravitational field because it does not recognize any distinction between fields. This is principally a matter of semantics. To correct the confusion we designate a new tensor $\Theta^{\mu\nu}$ to be the energy-momentum tensor of the gravitating source. The field equation is then

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\Theta^{\mu\nu}. \tag{19}$$

We examine the case where there is no gravitating source, $\Theta^{\mu\nu} = 0$. We want to prove

$$\Theta^{\mu\nu} = 0 \rightarrow R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0. \tag{20}$$

1. Einstein's field equation for $\Theta^{\mu\nu} = 0$

Einstein's field equation minimizes the action S_{gr} of general relativity where

$$S_{gr} = \int R\sqrt{-g}d^4x \tag{21}$$

The R is again the scalar curvature and here $g = det(g^{ij})$. This action can be minimized by taking the boundary conditions to be at a fixed geodesic distance and then minimizing the scalar curvature, R, on the volume.

In the action of knot physics, $S_k = \int \sigma + F^{\mu\nu}F_{\mu\nu}dM$. If $\Theta^{\mu\nu} = 0$ then the fields are zero, therefore $A^\nu = x^\nu$, and $S_k = \int \sigma dM$, which is proportional to the volume. The boundary is fixed in the Minkowski 6-space. Then the volume is minimized by changing the shape of the manifold. We would like to show that minimizing volume implies minimizing S_{gr}. Intuitively, we are showing that removing bumps (positive scalar curvature) reduces volume.

To show this we can take an infinitesimal region W and show that if the shape can be changed to reduce R on that region, then that change also reduces the volume. Let p be a point at the center of W and let B be the boundary of W. We assume that B is fixed and we want to minimize the volume of the interior. We use the exponential map beginning at p which gives expanding radii along geodesics emanating from p. At each radius there is a ball $B(r)$ which is the endpoints of the geodesics at radius r. The exponential map reaches the boundary at radius r_f and then $B(r_f) = B$ and $area(B(r_f)) = area(B)$. The volume of this ball is given by the integral over the shells $B(r)$:

$$V(W) = \int_0^{r_f} area(B(r))dr \tag{22}$$

We now allow the interior of W to change shape and observe the affect on the volume $V(W)$. The ending radius r_f is not fixed. However, the ending area is fixed; $area(B(r_f)) = area(B)$. Therefore we minimize $V(W) = \int_0^{r_f} area(B(r))dr$ by making the function $area(B(r))$ as convex as possible as a function of r. This implies maximizing the coefficients of r of degree 2 or higher.

From Riemannian geometry we know that

$$area(B(r)) = (1 - \frac{R}{6n}r^2)A(r) + ... \tag{23}$$

where R is the scalar curvature, n is the dimension of the manifold, and A is the area of the same radius ball in flat space. The coefficient of the second degree term of r is $-\frac{R}{6n}$. Thus we make the function more convex by minimizing R. Therefore, reducing the volume implies reducing R. Therefore, optimizing $S_k = \int \sigma dM$ implies minimizing $S_{gr} = \int R\sqrt{-g}d^4x$. From this, we see that the Einstein field equation holds on our manifold M for $\Theta^{\mu\nu} = 0$.

2. Einstein's field equation for arbitrary $\Theta^{\mu\nu}$

We still have two problems with Einstein's field equation. If $\Theta^{\mu\nu}$ is not the energy-momentum tensor, then what is it? Additionally, the curvature tensor and the metric are 6-dimensional objects. We cannot recover Einstein's field equation if we have to use information that is beyond the scope of our measurements. The two problems negate each other: the tensor $\Theta^{\mu\nu}$ is the remaining geometry when we force the equation into four dimensions.

To conform to the geometry of general relativity we assume a 4-dimensional manifold, call it M_4, which is geometrically flat (meaning that it occupies a four-dimensional linear subspace of the 6-space) and we assume that the manifold M asymptotically approaches M_4. We will assume two metrics on our subspace M_4. Call the first one $f^{\mu\nu}$ and let it be the Minkowski distance between two points on the subspace. If M is parallel to the subspace then $f^{\mu\nu} = \bar{\eta}^{\mu\nu}$. Call the second metric $h^{\mu\nu}$ and let it generate the geodesics on M_4. We choose $h^{\mu\nu}$ so that the geodesics on M project onto the geodesics on M_4. Using $h^{\mu\nu}$ we can define a Ricci curvature $H^{\mu\nu}$ and its scalar curvature H. Now we have a more appropriate setting for $\Theta^{\mu\nu}$. We define $\Theta^{\mu\nu}$ by

$$H^{\mu\nu} - \frac{1}{2}h^{\mu\nu}H = -\Theta^{\mu\nu}. \tag{24}$$

If the geodesics on M are approximately the same as on M_4 then we have the implication

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \rightarrow H^{\mu\nu} - \frac{1}{2}h^{\mu\nu}H = -\Theta^{\mu\nu} = 0. \tag{25}$$

For the case $\Theta^{\mu\nu} \neq 0$, we examine the case where the metric $h^{\mu\nu}$ approaches flatness. We would like to establish the relationship between $\Theta^{\mu\nu}$ and $T^{\mu\nu}$ as $h^{\mu\nu}$ approaches $f^{\mu\nu}$. The Bianchi identity is

$$-\nabla_\mu \Theta^{\mu\nu} = \nabla_\mu (H^{\mu\nu} - \frac{1}{2}h^{\mu\nu}H) = 0 \tag{26}$$

where we assume that the covariant derivative is taken with respect to the metric $h^{\mu\nu}$. As $h^{\mu\nu}$ approaches $f^{\mu\nu}$, $\nabla_\mu \Theta^{\mu\nu}$ approaches $\partial_\mu \Theta^{\mu\nu}$, which implies $\partial_\mu \Theta^{\mu\nu} \rightarrow 0$. Therefore $\Theta^{\mu\nu}$ is a conserved tensor in the limit. $H^{\mu\nu}$ depends only on $h^{\mu\nu}$ therefore $\Theta^{\mu\nu}$ should depend only on $h^{\mu\nu}$. The only conserved tensors of our metric are $T^{\mu\nu}$ and the charge current j^ν. By inspection we see that the charge current makes no contribution. Therefore $\Theta^{\mu\nu}$ must depend on $T^{\mu\nu}$. Further, we know that if $T^{\mu\nu} = h^{\mu\nu}$ then $\Theta^{\mu\nu} = 0$. Therefore, in the limit as $h^{\mu\nu}$ approaches $f^{\mu\nu}$ we have $\Theta^{\mu\nu} = k(T^{\mu\nu} - h^{\mu\nu})$ for some constant k. Returning to Einstein's field equation we see that $k = 8\pi$. Thus, in the limit

$$\Theta^{\mu\nu} = 8\pi(T^{\mu\nu} - h^{\mu\nu}). \tag{27}$$

If there are no particles and no electroweak fields then $\Theta^{\mu\nu}$ is zero. If there are particles or electroweak fields then $\Theta^{\mu\nu}$ is approximately proportional to their energy-momentum minus the energy-momentum of flat spacetime. This matches our expectations for a gravitational source tensor.

From our derivation we see that the manifold geometry resembles general relativity for cases like gravitating bodies and gravitational waves. However, for non-flat cases inside particle knots or large-scale spacetime distortion the difference from general relativity can be large.

B. Geometry of gravity

We need to find a geometry that leads to curvature in a gravitational field. Particles rotate. That rotation generates waves. The curvature caused by those waves is shown by the field equations above. Those waves are the gravitons of knot physics.

The geometry determines the degrees of freedom of the gravitational field. For a weak field, the six coordinates of a point on the manifold $(p_0, p_1, p_2, p_3, f_1, f_2)$ can be described by two scalar functions of p_i, in this case f_1 and f_2. Therefore the gravitational field has two degrees of freedom in this description. This is equal to the degrees of freedom of a linear spin-2 field in flat spacetime, which describes general relativity for weak fields. Thus the degrees of freedom for general relativity and knot physics are the same for weak fields.

A single particle produces a wave of the form $(t, p_1, p_2, p_3, b_1 cos(kt), b_2 sin(kt))$ for some constants k, b_1, and b_2. Depending on k, b_1, and b_2, the rotation in the last two coordinates can either be in the negative direction or the positive direction (clockwise or counterclockwise). A collection of particles produces a collection of waves that combine. Each of these contributing waves can likewise rotate in the positive direction or the negative direction. The Lagrangian for a gravitational field optimizes the action $S_k = \int \sigma dM$. This Lagrangian is optimized when all of the contributing waves have the same rotational direction, either all positive or all negative. The geometry of the gravitational field is therefore bistable and settles into one of two orientations. This bistability breaks symmetry. It is possible that the entire observable universe has the same orientation. The broken symmetry may explain the broken P and CP symmetries of particle physics. This would require a coupling of gravitational geometry to particle geometry.

VIII. QUANTUM FIELD THEORY

We would like to explain quantum phenomena in a geometric context. Quantum phenomena require non-determinism, which requires multiple possible results for a particular initial state. The double-slit experiment shows that those possibilities interfere with each other. We will make a series of changes to our assumptions to produce results that are consistent with quantum field theory.

A. Interference

We want to replace the spacetime manifold M with a set of branching and recombining manifolds, which we will call *paths*. Those paths will contribute to the sum-over-histories path integral of quantum field theory. As an example, we will assume that M branches only once and then recombines. We will say that there are two paths φ_1 and φ_2. Each path is a 4-manifold with vector field A^ν and $\partial_\nu A^\nu = -2$. The paths φ_1 and φ_2 are the same except on some compact subset of each. From a causal perspective, the paths branch and then recombine.

We note that two paths φ_1 and φ_2 can pass through each other. To do that they first recombine along some intersection and then branch again on the other side. However, we still require that each path φ cannot pass through itself.

We gave an example of a pair of branching paths φ_1 and φ_2 that branch and recombine. We can extend that branching to a larger but finite number of paths that branch and recombine. We want to define integration over the paths. For the two path example, we get the action integral by integrating once where φ_1 and φ_2 are the same and also once each over the places where they are distinct. If φ_1 and φ_2 are everywhere the same then the integral is just over one manifold. If they are everywhere distinct then the integral is over two manifolds. We will continue to use M to denote the set of paths. Then integrating a function f over M means integrating over all of the paths, denoted $\int_M f dM$. We can now modify our assumptions.

We assume that the action $\int_M \sigma dM$ equals some fixed constant for some constant σ. We will later show how this relates to the Lagrangian.

We assume that the curvature of every path is everywhere less than some

finite bound.

Suppose we have at least two paths φ_1 and φ_2 that extend to some boundary. The two paths can be connected by other paths. Assume that φ_1 and φ_2 are flat and parallel. The fixed total action constraint sets a bound on how many connecting paths can go between the two. If φ_1 and φ_2 are close then many connecting paths are possible. If they are far then few connecting paths are possible. Entropically, this favors φ_1 and φ_2 being close. However, φ_1 and φ_2 are not necessarily flat or parallel. With bounded curvature, if φ_1 and φ_2 are much closer than their curvature bounds then the two paths have approximately the same geometry, which is entropically less likely. Therefore we would expect that the two paths would most likely remain at an average distance that is bounded both above and below. In general, with fixed total action we can have a large number of paths. We would expect the gaps between those paths to be roughly the size of the curvature bound. In a reference frame where the paths are at rest on average, at each point the total set of paths has some diameter, the smallest 5-sphere that contains every path. The expected diameter is determined by the number of paths and the curvature bound. The actual diameter of the path set has some variance around the expected diameter.

Now suppose we have two paths φ_1 and φ_2 that are close at time t_a and then recombine at time t_b. After recombination each point of φ_1 and φ_2 match. Suppose that at time t_a the path slices are $\varphi_i(t_a)$ for each path and they recombine to $\varphi_{sum}(t_b)$ for each path. If the paths are approximately at rest then the most probable recombination is that the points move to the average. Then the recombination would produce $\varphi_{sum}(t_b) = (1/2)(\varphi_1(t_a) + \varphi_2(t_a))$. We will later use this recombination to explain the interference of quantum wave-functions.

B. Quantum manifolds with a Lagrangian

To include the Lagrangian on paths, we required that the action $\int_M \sigma dM$ is equal to some fixed constant. Then probability produces a result similar to optimization, as we show here.

Consider a curve φ_0 on a plane that wanders at random with bounded curvature. We assume that the curve has a fixed action (its length) equal to S_0. This random curve is similar to a random walk. If the end points of the curve are fixed at points A and C then the entropy of the curve is maximal when the curve is close to the straight line from A to C. To see this, one can imagine the possibility that the curve is stretched as tight as possible to go from A to some point B and from B to C. The condition of being stretched tight would reduce the entropy to nearly zero. In the limit that the curvature of φ_0 goes to infinity, φ_0 is within zero distance of the straight line from A to C with probability one. Call that straight line Φ_{max}. (The curve Φ_{max} is not required to have the same length S_0 as the random curve.) Now consider multiple paths φ from A to C that branch and recombine. Each path contributes to the total action S. The paths will remain close to Φ_{max} because of probability. However, the curvature bound pushes each path to have some minimum difference and we therefore obtain a set of paths centered around the straight line curve Φ_{max}. The diameter of the set of paths depends on the total action S and the curvature bound.

Increasing the dimension, we can consider a random 2-dimensional manifold φ_0 with bounded curvature embedded in a 3-dimensional space. We specify that the surface φ_0 has fixed action S_0, which is its area. In this case the boundary condition is a one-dimensional manifold that is the boundary of the surface. The entropy is maximized when the path φ_0 is as close as possible to Φ_{max}, the minimal surface that has the same boundary. If we allowed the curvature of φ_0 to be infinite then φ_0 would be within zero distance of the minimal surface Φ_{max} with probability one. Now we can again invoke the quantum condition and add more paths. These additional paths will center around Φ_{max} while varying around it up to some diameter. The diameter again depends on the total action S and the curvature bound.

C. The spacetime manifold

Next we consider φ_0, a random 4-manifold in a Minkowski 6-space. We use the constant action constraint such that $\int \sigma dM$ is constant when integrated over φ_0. With no further constraints this would imply that φ_0 would be close to the minimal volume manifold. However, we want to allow the possibility that points can be displaced from rest. To do that we have an A^ν field on each path. Let $A^\nu(p)$ be the rest position of p. If the point is at its rest position then $A^\nu(p) = x^\nu(p)$. If a point is displaced from rest then $A^\nu(p) \neq x^\nu(p)$. Each path has a vector field A^ν that describes point displacement and we require $\partial_\nu A^\nu = -2$.

We can use A^ν to describe a causality condition in a particular frame. We require that the set of points that are in a time slice $t = constant$ in their rest position must always be at spacelike separation from each other. For displaced points this means the set $A^0 =$

constant must be a spacelike slice. Extending to general frames, this means that the sets $A^\nu t_\nu = constant$ must be spacelike for any timelike vector t_ν. This constraint means the gradient $A^{0,\mu}$ affects the set of possible paths. The spacelike components of the gradient $A^{0,\mu}$ can have any magnitude less than or equal to 1. How are path probabilities affected?

We can consider a simpler case with a flat 2-dimensional manifold in a Minkowski space. There is a path φ, which is 2-dimensional in this case, and φ has an A^ν value at each point. Take a slice of that path with $A^0 = constant$. Take a piece of that slice that begins at $x^\nu = (0,0)$ and goes to $x^\nu = (D,1)$ for some number D between 1 and -1. For simplicity we can assume the slice is a random walk with N segments and each segment can have $A^{0,1}$ gradient +1 or -1. The average $A^{0,1}$ gradient is D. If D is 1 or -1 then the number of random walks is one. Let $D = 1 - 2m/N$ with $0 \leq m \leq N$. Then there are m places on the random walk where the slope is -1. Then there are $\frac{N!}{m!(N-m)!}$ possibilities. The number of possibilities is maximized for $D = 0$, for which $m = N/2$. If the number of path possibilities is P then the Lagrangian is $L = -ln(P)$. Then the Lagrangian as a function of m is

$$L(m) = -ln(\frac{N!}{m!(N-m)!}) = -ln(N!)+ln(m!)+ln(N-m)! = -\sum_{a=1}^{N} ln(a)+\sum_{b=1}^{m} ln(b)+\sum_{c=1}^{N-m} ln(c) \tag{28}$$

and taking the continuous version of the discrete functions,

$$L(m) = \int_1^N ln(a)da + \int_1^m ln(b)db + \int_1^{N-m} ln(c)dc \tag{29}$$

then taking the Taylor series expansion of $L(m)$ around $m = N/2$.

$$L(m) = L(N/2) + L'(N/2)(m - N/2) + (1/2)L''(N/2)(m - N/2)^2 + ... \tag{30}$$

looking at the linear term,

$$L'(N/2) = ln(N/2) - ln(N - N/2) = 0 \tag{31}$$

and the quadratic term

$$L''(N/2) = (N/2)^{-1} + (N - N/2)^{-1} = 4/N \neq 0 \tag{32}$$

Therefore the Lagrangian is a constant, plus a quadratic term, plus higher order terms. We see that the Lagrangian depends on the slope of the $A^0 = constant$ curve, which implies a dependence on $A^{0,\mu}$. Now we want to show that it depends on $A^{0,\mu} - A^{\mu,0}$.

There are two ways of changing the mapping of points to locations. One way is the φ mapping, which affects the A^{ν} vector field. Another way is Lorentz transformations. The φ mapping may affect path probabilities but Lorentz transformations should not. Assume there is a φ mapping with vector field A^{ν} and derivatives $A^{\nu,\mu}$. Then we can decompose the mapping as a Lorentz transformation and a displacement of points. A Lorentz boost is a symmetric transformation. If the transformation was entirely Lorentz boost then $A^{\nu,0} = A^{0,\nu}$. The amount of anti-symmetry $A^{\nu,0} - A^{0,\nu}$ therefore determines the amount that the paths are constrained. The quadratic term is $(A^{0,\nu} - A^{\nu,0})(A_{0,\nu} - A_{\nu,0})$. There is a unique extension to the Lorentz invariant $(A^{\mu,\nu} - A^{\nu,\mu})(A_{\mu,\nu} - A_{\nu,\mu}) = F^{\mu\nu}F_{\mu\nu}$. We see that path probabilities are maximized around the path that optimizes $\int_M \sigma + F^{\mu\nu}F_{\mu\nu} dM$.

Therefore we can use $\sigma + F^{\mu\nu}F_{\mu\nu}$ as the Lagrangian to find Φ_{max}. The quantum condition produces a set of paths that are centered around Φ_{max}. The diameter of the set again depends on the total action and the curvature bound.

D. Collapse of state

The set of paths can allow for multiple simultaneous possibilities, but the set of paths is finite. When branching possibilities extend beyond the number of possible paths, the state of the system must collapse, as in Fig. 1. When that happens, other paths that are too different from the collapsed state have to pull tight and rejoin with the paths corresponding to the collapsed state.

E. Recombination

How do we decide whether two paths can recombine? First the two paths must have the same topology. In [1] we describe particle topology. For the moment we simply note that distinct topologies cannot recombine because the points in each path will not match. Therefore, if one path has flat space at a particular location and another path has a particle, then the two cannot recombine. If two paths have the same particle at the same location, can they recombine? The two particles can still have different phases. Combining the two paths would take the average of the two paths. In [1] we show that the particle's phase corresponds to its rotation in the x_4 and x_5 axes. A point on a particle in path φ_i has

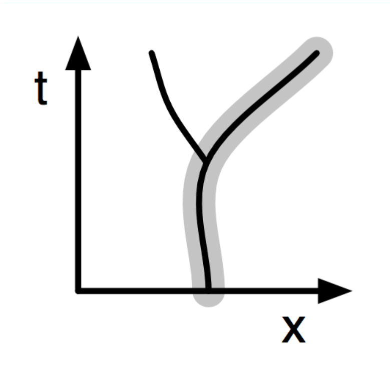

FIG. 1: This diagram illustrates a collapse of state. The dark lines represent possible paths. For example, the dark lines could correspond to measuring a particle spin as being either spin up or spin down. After the time of measurement the paths diverge because the measured spin state has an effect on the machine that measured it. The effect on the machine leads to paths that become increasingly different. Because these paths expand beyond the set of possible paths (the gray region) the state of the system collapses to one state or the other. The t coordinate here represents time. The x coordinate is just a generalized spacelike coordinate representing the geometry of the path at that time.

location

$$\varphi_i : p \rightarrow (t, x, y, z, cos(\theta_i), sin(\theta_i)) \tag{33}$$

so that the x_4, x_5 coordinates can be written as a complex number $e^{i\theta}$. Recombining two paths maps a point p to the average,

$$(1/2)(\varphi_1 + \varphi_2) : p \rightarrow (t, x, y, z, (1/2)[cos(\theta_1) + cos(\theta_2)], (1/2)[sin(\theta_1) + sin(\theta_2)]) \tag{34}$$

which would be $(1/2)[e^{i\theta_1} + e^{i\theta_2}]$. For a knot, this shrinks the knot's radius along two axes. The probability of a particular physical measurement depends on the number of paths that correspond to that measurement. The number of paths that correspond to a particular knot geometry depends on the size of the knot. To compare, consider the probability of a random 1-dimensional path in 3-space that passes through a circle of radius r. The probability of passing through that circle depends on the square of the radius, r^2. Likewise, for a knot, the number of paths consistent with the knot depends on the "interior volume" of the knot. While there is no space that is actually inside a knot, scaling the knot geometry linearly will likewise scale the interior volume by the same amount. For a knot that is rotating in x_4 and x_5, two of its axes will scale because of the recombination. Writing the knot phase as $\phi = re^{i\theta}$, the probability of observing a knot with a particular phase is proportional to $|\phi|^2 = r^2$. If there are M paths that recombine and have a knot at coordinates x then the summed knot has phase $\phi_{sum}(x) = (1/M)\sum_{i=1}^{M}\phi_i(x)$. Then the probability is $P = |\phi_{sum}(x)|^2$. We can use one complex scalar to represent both the knot phase and the fraction of the paths that have a knot at that location. Assume that there are N total paths and M of those paths have a knot at x. Then let $\phi_i(x)$ be the phase of those knots at x and $\phi_i(x) = 0$ if there is no knot at x in path i. We want ψ such that the phase angle of ψ is the phase angle of the recombined knot and $|\psi(x)|^2$ is the probability of a observing the knot at x. The sum over paths with knots is $\phi(x) = (1/M)\sum_{i=1}^{N}\phi_i(x)$. The total probability should be $P = (M/N)|\phi|^2$. Therefore the sum should be

$$\psi(x) = (M/N)^{1/2}(1/M)\sum_{i=1}^{N}\phi_i(x) = (1/MN)^{1/2}\sum_{i=1}^{N}\phi_i(x) \tag{35}$$

The function ψ uses the angle θ to describe the knot's phase angle. The magnitude of ψ describes both the knot's radius and the fraction of the paths that have a knot there. After recombination the knot returns to its equilibrium radius and the only remaining effects of recombination are the phase angle and the fraction of paths with knots at x.

F. The path integral

The complex phase $\psi = re^{i\theta}$ accounts for particle phase and the addition of ψ functions performs the recombination of paths. Rather than considering the geometry of every path, we can combine all the paths as functions of a complex variable.

Now the method is exactly the same as in quantum field theory. In quantum field theory we use the Hilbert space basis of position states $|q\rangle$ and the Hamiltonian operator H to find the probability amplitude to transition from q_1 to q_N. To avoid confusion we define two Hamiltonians, H_q and H_k such that H_q is the Hamiltonian of quantum field theory and H_k is the Hamiltonian that generates knot physics. To find H_k we begin with $S = \int \sigma + F^{\mu\nu}F_{\mu\nu}dM$. That action uniquely determines a Hamiltonian. While the Lagrangian is relatively simple, the boundary conditions associated with non-self-intersection of the manifold are complicated. This complexity leads to complexity in the Hamiltonian H_k.

Using H_k,

$$\langle q_N|e^{-iH_kT}|q_1\rangle = (\prod_i \int dq_j)\langle q_N|e^{-iH_k\Delta t}|q_{N-1}\rangle ... \langle q_2|e^{-iH_k\Delta t}|q_1\rangle \tag{36}$$

It remains to show that H_k and H_q give the same results. This problem is non-trivial. The previous sections on the electroweak force, the strong force, and gravity show that H_k is similar to H_q for some important cases.

G. Branching and Planck's constant

How does this relate to Planck's constant h? Paths recombine if they are close enough. By our description, a difference of particle phase is close whereas large differences of particle position are not close. Suppose we have two paths that differ in the amount of action in some region. How much difference is required before the paths are no longer considered close? In Minkowski space, photons have zero length. From photon energies we see that a difference of the action of amount h is adequate to separate two paths.

H. A comment on the terms

We have been using two different methods of describing spacetime. In one method we refer to the spacetime manifold, M. In the other method we refer to paths, φ, which contribute to the probability of a transition in the sense of the sum-over-histories approach to quantum field theory. In general the path description always applies. Transition probabilities are always generated by the path possibilities. However, there are topics for which the set of paths is not the relevant feature. For those purposes we want to discuss what is possible for any manifold having the features of the spacetime manifold. Then we will refer to the spacetime manifold, M, with the implicit statement that the properties which must hold for M also must hold for any path φ.

IX. COSMOLOGY

Cosmology describes the expansion of the universe. Particles are knots in spacetime and many varieties of particle interaction release some of that bound space. This released spacetime may account for some of the expansion of the universe. If that is the case, then the expansion is non-uniform. Bound spacetime is released most rapidly where energy is released most rapidly. However, the geometry of spacetime cannot absorb that excess instantly across all space. Releasing that space would produce a positive scalar curvature R. This would have the same gravitational effect as a nonzero energy-momentum trace, T^{α}_{α}, but without a massive source. These effects would be most apparent in regions that had released significant quantities of bound space, for example old galaxies and galactic clusters. This is similar to the gravitational effect that is allocated to dark matter– a variety of matter that interacts only gravitationally and has never been directly observed. The contribution of this geometric distortion may reduce or eliminate the need for a dark matter explanation.

X. CONCLUSION

Our purpose was to use geometric methods to develop the results of physics. We found that particle topology requires a 4-dimensional manifold M embedded in a 6-dimensional Minkowski space. The electroweak field requires a potential function A^{ν} on M that is distinct from the coordinate x^{ν}. Quantum phenomena require non-determinism and interference, which implies branching paths. Using these assumptions we can reproduce many of the results of physics within a coherent theory.

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